A SOLUTION OF THE TIME-DEPENDENT ADVECTION-DIFFUSION EQUATION

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1. Introduction

In the last decades, great progress was made to the issue of searching analytical solutions for the steady-state advection-diffusion equation in order to simulate the pollutant dispersion in the Planetary Boundary Layer (PBL). The solutions were valid only for very specialized practical situations with restrictions on wind and eddy diffusivities vertical profiles. Recently appeared an analytical solution of the steady-state advection-diffusion equation, applying the new GILTT method (Generalized Integral Laplace Transform Technique) that accepts any wind and eddy diffusivity vertical profile.The main idea of this methodology comprehends the following steps: expansion of the concentration in series of eigenfunctions attained from an auxiliary problem, replacing this equation in the advection-diffusion equation and taking moments, we come out with a matrix ordinary differential equation that is solved analytically by the Laplace Transform technique. For more information, see Moreira et al. (2009). In this paper we extend these last results presenting a time-dependent solution.

2. The Time-Dependent Solution

The advection-diffusion equation is a deterministic approach to dispersion of pollutants in the atmosphere. It is obtained by mass conservation combined with first order closure (K-Theory). Here $\overline{c}$ denotes the mean concentration of a passive contaminant; $\overline{u}$, $\overline{v}$ and $\overline{w}$ are the cartesian components of the mean wind speed in the direction x, y and z, respectively; k_x, k_y and k_z are the eddy diffusivities. For brevity, here, we align the predominant wind direction with the direction of the x coordinate, $\overline{v} = \overline{w} = 0$, and the advection-diffusion equation simplifies to:

$$\frac{\partial \overline{c}}{\partial t} + \overline{u}\frac{\partial \overline{c}}{\partial x} = \frac{\partial}{\partial x}\left(k_x \frac{\partial \overline{c}}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial \overline{c}}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial \overline{c}}{\partial z}\right), \tag{1}$$

The space-time domain is given by $t > 0$, $0 < x < L_x$, $0 < y < L_y$, $0 < z < z_i$ and the equation (1) is subject to the following boundary and initial conditions $k_x \frac{\partial \overline{c}}{\partial x} = 0$ in $x = 0$ and $x = L_x$, $k_y \frac{\partial \overline{c}}{\partial y} = 0$ in $y = 0$ and $y = L_y$, $k_z \frac{\partial \overline{c}}{\partial z} = 0$ in $z = 0$ and $z = z_i$, $\overline{c}(x, y, z, 0) = 0$. Moreover, the source condition is $\overline{u}\overline{c}(0, 0, z, t) = q\delta(z - H_s)\delta(t - t_0)$.
In order to construct a time-dependent solution, first we separate time-dependent wind field and time-dependent eddy diffusivity contributions from their time averaged values:

$$\overline{u}(z,t) = \overline{U}_x(z) + U(z,t) \tag{2a}$$
$$k_x(z,t) = \overline{K}_x(z) + K_x(z,t) \tag{2b}$$
$$k_y(z,t) = \overline{K}_y(z) + K_y(z,t) \tag{2c}$$
$$k_z(z,t) = \overline{K}_z(z) + K_z(z,t) \tag{2d}$$

Where $\overline{U}(z)$, $\overline{K}_x(z)$, $\overline{K}_y(z)$ and $\overline{K}_z(z)$ are the time averages. Upon inserting these assumptions in equation (1) yields:

$$\frac{\partial \overline{C}}{\partial t} + \overline{U}\frac{\partial \overline{C}}{\partial x} - \frac{\partial}{\partial x}\left(\overline{K}_x \frac{\partial \overline{C}}{\partial x}\right) - \frac{\partial}{\partial y}\left(\overline{K}_y \frac{\partial \overline{C}}{\partial y}\right) - \frac{\partial}{\partial z}\left(\overline{K}_z \frac{\partial \overline{C}}{\partial z}\right) = -U\frac{\partial \overline{C}}{\partial x} + \frac{\partial}{\partial x}\left(K_x \frac{\partial \overline{C}}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial \overline{C}}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial \overline{C}}{\partial z}\right). \tag{3}$$

According to the idea of the decomposition method (Adominian, 1988) the solution of (3) is written as a truncated expansion:

$$\overline{c}(x,y,z,t) = \sum_{l=0}^{J} \overline{C}_l(x,y,z,t). \tag{4}$$

These new degrees of freedom for each component may now be used to decompose (3) into a set of advection-diffusion equations, that together form a recursive scheme:

$$\begin{cases} \frac{\partial \overline{C}_0}{\partial t} + \overline{U}\frac{\partial \overline{C}_0}{\partial x} - \frac{\partial}{\partial x}\left(\overline{K}_x \frac{\partial \overline{C}_0}{\partial x}\right) - \frac{\partial}{\partial y}\left(\overline{K}_y \frac{\partial \overline{C}_0}{\partial y}\right) - \frac{\partial}{\partial z}\left(\overline{K}_z \frac{\partial \overline{C}_0}{\partial z}\right) = 0 \\ \frac{\partial \overline{C}_1}{\partial t} + \overline{U}\frac{\partial \overline{C}_1}{\partial x} - \frac{\partial}{\partial x}\left(\overline{K}_x \frac{\partial \overline{C}_1}{\partial x}\right) - \frac{\partial}{\partial y}\left(\overline{K}_y \frac{\partial \overline{C}_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\overline{K}_z \frac{\partial \overline{C}_1}{\partial z}\right) = S_0 \\ \vdots \\ \frac{\partial \overline{C}_l}{\partial t} + \overline{U}\frac{\partial \overline{C}_l}{\partial x}\frac{\partial}{\partial x}\left(\overline{K}_x \frac{\partial \overline{C}_l}{\partial x}\right) - \frac{\partial}{\partial y}\left(\overline{K}_y \frac{\partial \overline{C}_l}{\partial y}\right) - \frac{\partial}{\partial z}\left(\overline{K}_z \frac{\partial \overline{C}_l}{\partial z}\right) = S_{l-1} \end{cases} \tag{5}$$

The above resulting recursive system is solved analytically by the GILTT method.

3. Preliminary Comparison with Experimental Data

For a prelimary comparison with experimental data we used the OLAD dataset (Biltoft et al., 1999). In particular, we used the dataset of the 12th September 1997, where sulfurhexafluoride (SF6) was released by a truck mounted disseminator at 3 m above ground level and following the Bravo route for 10 km. The beginning of the emission was at 6 hours and 58 minutes with duration of 10 minutes. According to Chang et al. (2001) this experiment has the characteristic of low wind speed (less or equal 3.5 m/s). Further, the planetary boundary layer showed a stable condition during the sampling period. The pollution concentrations where measured by fifteen analysers (LC101-LC115) located along the route Foxtrot at a distance of 2 km "parallel" to the Bravo Route The whole-air samplers produced time-averaged (15-min) tracer gas concentrations.
While the real experiment used a continuous line source to dissemination the tracer, we used in the simulations a finite number of point source to represent the line source. We used ten points source and in each source the release duration was Δt. Thus the concentration in each sampler is defined by:

$$C_l(x,y,z,t) = \sum_{a=0}^{9} c_l(x,y,z,t - a\Delta t - \frac{\Delta t}{2})$$

where *a* denotes the displacement centred line segment source.
The dataset utilized presents stable condition only. For a stable PBL we used eddy diffusion parameterization proposed by Degrazia et al. (2000):

$$K_z = \frac{0.3(1 - z/z_i)u_* z}{1 + 3.7(z/\Lambda)} \tag{6}$$

where $\Lambda = L(1 - z/z_i)^{5/4}$, whereas Degrazia et al. (1996) proposed for a stable boundary layer an algebraic formulation for the eddy diffusivity in the x and y direction according to :

$$\frac{K_\alpha}{u_* z_i} = \frac{2\sqrt{\pi}0.64a_\nu^2(1 - \frac{z}{z_i})^{\alpha_1}(\frac{z}{z_i})X'[2\sqrt{\pi}0.64a_\nu^2(\frac{z}{z_i}) + 8a_\nu(f_m)_\nu(1 - \frac{z}{z_i})^{\alpha_1/2}X']}{[2\sqrt{\pi}0.64(\frac{z}{z_i}) + 16a_\nu(f_m)_\nu(1 - \frac{z}{z_i})^{\alpha_1/2}X']^2} \tag{7}$$

where $(f_m)_\nu = (f_m)_{n,\nu}(1 + 3.7(z/\Lambda))$ is the frequency of the spectral peak, $(f_m)_{n,\nu} = 0.33$ is the frequency of the spectral peak in the neutral stratification (Sorbjan, 1989), $\Lambda = L(1 - z/z_i)^{(1.5\alpha_1 - \alpha_2)}$ ($\alpha_1 = \frac{3}{2}$; $\alpha_2 = 1$) is the local Monin-Obukhov length, $a_\nu = (2.7C_\nu)^{\frac{1}{2}}/(f_m)_{n,\nu}^{\frac{1}{3}}$, where $C_\nu = 0.4$, u_* is the friction velocity and $X' = xu_*/\overline{u}z_i$ represents the non-dimensional distance.
The wind speed profile can be described by the power law (Irwin, 1979)

$$\frac{\overline{u}}{\overline{u_1}} = \left(\frac{z}{z_1}\right)^n \tag{8}$$

4. Results

We used the experiment five of OLAD dataset. We used the meteorological data presented in Table 1. They were calculated using the observed meteorological data in Degrazia (2005). The data u, u_*, L and z_i represent the wind speed in 10 meters, the friction velocity, Monin-Obukhov length and the height of the planetary boundary layer, respectively.
The statistical indices (normalized mean square error (NMSE), correlation coefficient (COR) and fractional standard deviations (FS)) are shown in Table 2.
In Figure 1 it is shown the scatter plotter of observed and computed data.

Table 1: *Meteorological conditions of OLAD 5 (Degrazia, 2005)*

Olad 5	$u(10m)$ (ms^{-1})	u_* (ms^{-1})	L (m)	z_i (m)
6:45 - 7:00	1,71	0,10	43,48	171,60
7:00 - 7:15	1,95	0,12	64,81	223,80
7:15 - 7:30	1,82	0,11	127,27	303,03
7:30 - 7:45	1,90	0,11	221,12	407,32

Table 2: *Statistical comparison between observed and predict concentration*

Experimento	NMSE	COR	FS
OLAD 5/Foxtrot	0.14	0.80	0.64

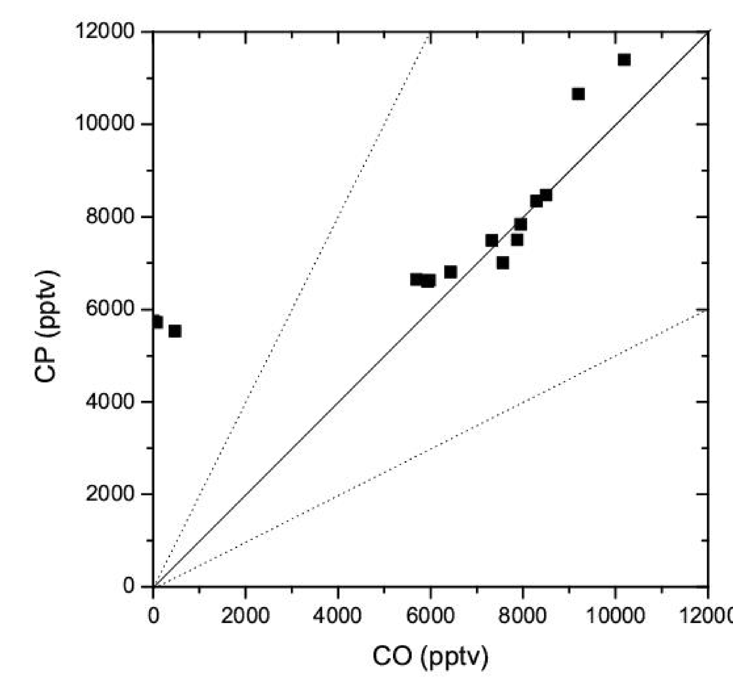

Figure 1: *Observed (CO) and predict (CP) scatter plot*

5. Conclusions

A general solution of the steady-state advection-diffusion equation with any restriction on wind and eddy diffusivity vertical profiles is known in the literature as the GILTT approach (Tirabassi et al., 2008; Moreira et al., 2009). We extend these last results and we present a time-dependent solution.
The advection-diffusion equation was solved by a combination of a decomposition method, and the GILTT approach. The first part of the solution method produces a recursive set of equations, where each of the equations have a known solution by GILTT. We also analysed stability of the procedure (not included here) which showed that only a small recursion depth is necessary in order attain an acceptable accuracy.
A comparison with the OLAD experimental dataset was presented.

6. References

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