THE GROUND LEVEL CONCENTRATION FROM A POINT SOURCE

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Abstract: The Advection-Diffusion Equation (ADE) is solved for a constant pollutant emission from a point-like source placed inside an unstable Atmospheric Boundary Layer (ABL). The solution is obtained adopting the novel analytical approach named Generalized Integral Laplace Transform Technique (GILTT). The concentration solution of the equation is expressed through an infinite series expansion. After setting a realistic scenario through the wind and diffusivity parameterizations the Ground Level Concentration (GLC) is worked out, then an explicit approximate expression is provided for it allowing an analytic simple expression for the position and value of the maximum. Remarks arise on the ability to express value and position of the GLC as an explicit function of the parameters defining the ABL scenario and the source height.

Key words: *air pollution modelling, analytical solutions, advection-diffusion equation, maximum concentration, environmental impact assessment.*

1. INTRODUCTION

The analytical solution of the ADE has been performed following different approaches based on Gaussian and non-Gaussian solutions. Gaussian solutions represent a rather easy operative tool to be handled. Non-Gaussian analytical solutions represent a more realistic approach to represent atmospheric diffusion. Nonetheless using non-Gaussian approaches, solutions are much harder to be achieved, and often for rather simple parametrization profiles only. As an example, Demuth (1978) provided an analytical solution with power law parametrizations with the realistic assumption of a bounded ABL. Such a solution involves a series expansion of the concentration in terms of the Bessel functions. The solution has been implemented in the KAPPA-G model (Tirabassi et al. 1986). Then Lin and Hildemann (1997) extended the solution of Demuth (1978) with boundary conditions suitable for simulating dry deposition at the ground.

In this paper, a complete analytical solution of the steady-state ADE is presented. Such a solution is based on the GILTT method (Wortmann et al, 2005; Moreira et.al, 2005). The ability to handle the exact analytical solution allows to upgrade the study of the concentration, nonetheless due to the non-explicit dependence on the set of variables defining the ABL scenario and the source features, an explicit analytical approximation would represent a useful reference when application purposes are required.

2. THE GILTT SOLUTION

The two-dimensional steady-state ADE for an emitting point-like source in a stationary ABL is:

$$u(z)\frac{\partial C(x,z)}{\partial x} = \frac{\partial}{\partial z}\left(k_z(z)\frac{\partial C(x,z)}{\partial x}\right), \tag{1}$$

where along the x direction the longitudinal diffusion term has been neglected in respect to the advection term. In the above equation *C(x,z)* represents the cross-wind integrated three-dimensional time-independent concentration. *u(z)* is the horizontal mean wind and $k_z(z)$ is the vertical diffusivity, both are depending on the vertical coordinate *z*. The boundary conditions impose the flux to vanish at the extremes of the ABL (z=0,h), and the source condition is set to represent the point-like source placed at the height h_s above the ground level, namely:

$$u(z)C(0,z) = Q\delta(z-h_S), \tag{2}$$

where *Q* is the constant rate of emission and $\delta(z-h_s)$ is the Dirac δ function.

The GILTT technique provides a solution for Equation (1) which is written in terms of a converging infinite series expansion (Wortmann et al., 2005; Moreira et al., 2005)

$$C(x,z) = \sum_{i=0}^{\infty}\overline{c}_i(x)\phi_i(z), \tag{3}$$

where $\varphi_i(z)$ are the eigenfunctions solving the Sturm-Liouville equation, and $\overline{c}_i(x)$ are *x*-depending functions. Due to convergence the series can be truncated at a certain number *N* such that the rest $R_N(x,z)$ become negligible in respect of the partial sum. The results reported later are obtained after setting *N=190*, with a round off error limited below 0.5% (Tiesi et al. 2007).

3. TURBULENT PARAMETERIZATION

The choice of the turbulent parameterization is set to account for the dynamics processes occurring in the ABL. In the present paper the choice is addressed towards rather simple profiles, but still reasonably realistic. Moreover specific focus is devoted towards unstable regimes.

The choice of the vertical profile for the wind $u(z)$ is set to be following a power law (Panofsky and Dutton, 1988):

$$\frac{u(z)}{u_1} = \left(\frac{z}{z_1} \right)^{\alpha} \tag{4}$$

where u_1 is the mean wind velocity at the height z_1, while α is an exponent related to the turbulence intensity (Irwin, 1979). On the quantitative side, results will be provided setting $\alpha=0.1$, and the reference wind $u_1(0.01h)=3\ ms^{-1}$; these values are quite consistent with the whole range of unstable regimes pointed out by Pasquill and Smith (1984).

The vertical diffusivity parameterization is led by the K-theory assumption. According to Pleim and Chang (1992), for an unstable ABL it is defined as:

$$k_z(z) = k w_* z \left(1 - \frac{z}{h} \right) \tag{5}$$

where h is the height of the ABL, k is the von Karman constant which is set to 0.4, and w_* is the convective scaling parameter related to the Monin-Obukhov length L_{MO} and the mechanical friction parameter u_* as:

$$w_* = u_* \left(\frac{h}{L_{MO}} \right)^{1/3} \tag{6}$$

For convective scenarios L_{MO} is limited to values such that the relationship $\frac{h}{L_{MO}} < -10$ holds. Finally u_* is determined through similarity assumptions (Panofsly and Dutton, 1988; Zannetti, 1990).

4. GROUND LEVEL CONCENTRATION

From the solution of the ADE the GLC is obtained after setting $z=0$ inside the solution $C(x,z)$. Results will be reported in terms of the dimensionless GLC as follows:

$$C_{GLC}(x) = C(x,0) \frac{<u> h}{Q} \tag{7}$$

where $<u>$ is the vertically averaged wind introduced in Equation (4). The definition (7) has been introduced to obtain for any parameter choice $\lim_{x \to \infty} C_{GLC}(x) = 1$, according to the theoretical expectation for the two-dimensional ADE solution.

The scope of this paper is to provide a simple explicit expression for the maximum GLC $C_{MGLC}(x_M)$ occurring at the horizontal distance x_M as a function of the setting parameters for ABL scenario and source emission. Comparisons with data sets are reported in Moreira et al. (2006) and Buske et al. (2007).

Although the sum (3) represents the exact solution of the ADE (1) except for a round-off error, the series expansion misses manifest dependencies on ABL parameters and source height. On the other hand the main advantage of the GILTT technique is to allow to step from a differential-like approach, traditionally adopted to solve the ADE numerically, to a matrices algebra approach after applying the generalized Laplace transform. Then the core of the problem leads to investigate on the behaviour of the series (3) after setting $z=0$, and using the property of the Sturm-Liouville eigenfunction for which $\varphi_i(0)=1$ regardless the index i. Despite the choice of a profile depending approximation, there is a practical advantage in simplicity. In doing this, empirical parameters will appear and these are determined by fit procedures to best reproduce the exact solution. The dimensionless GLC defined in Equation (7) can be approximated as follows:

$$C_{GLC}(x) \approx \left[1 + \left(\frac{\kappa}{\lambda \tilde{x}} \right)^{c} \right]^{b} \exp \left[- \frac{\left(\pi \tilde{h}_S \right)^{1+2bc}}{\left(\lambda \tilde{x} \right)^{2bc}} \right] \tag{8}$$

where the tilded variables are meant to be normalized in respect to the ABL height h (e.g. $\tilde{x} = x/h$). Due to the negative values assumed by the Monin-Obukhov length, in the following it will be defined as the positive dimensionless parameter $\tilde{L}_{MO} = -L_{MO}/h$. Parameters b, c, κ and λ have been determined by fittings procedures on Equation (14) against the analytical solution and these are:

$$b = \tilde{h}_S^{5/2} + .17 \tag{9}$$

$$c = -5.48 \tilde{h}_S^{0.87} + 4.73 \tag{10}$$

$$\kappa = \left(\frac{\alpha + 1}{0.4277} \right)^{2.62} \tilde{h}_S^{0.41} \tag{11}$$

$$\lambda = \left(0.35 u_1 \right)^{-1} \left(\alpha + 1 \right)^{-1.3} w_* \tilde{h}_S^{.47} \tag{12}$$

It is easy to see the explicit dependency on the source height h_S, the wind parameters α and u_1, and the convection scaling parameter w_* which is related to the Monin-Obukhov length L_{MO} and the friction parameter u_* through the relationship (6).

The approximating expression for $C_{GLC}(x)$ is based on the study of the series expansion for $C(x,0)$. One the main features of the approximating function is that for short distances from the source height, instabilities are easily managed because the exponential function suppresses any divergence affecting the algebraic function. With the explicit approximation for $C_{GLC}(x)$ it is straightforward to evaluate its x-derivative and obtain the position

$$x_M^{2bc} = 2\frac{\left(\pi\tilde{h}_S\right)^{1+2bc}}{\lambda^{2bc}}e^{-\frac{1}{2}} \tag{13}$$

where a maximum for the GLC occurs

$$C_{MGLC}(x_M) = \left[1+\left(\frac{2\kappa}{\left(2\pi\tilde{h}_S\right)^{1+\frac{1}{2bc}}}\right)^c\right]^b e^{-\frac{1}{2}} \tag{14}$$

The expression for the position x_M is valid when the condition

$$\left(\frac{\kappa}{\lambda\tilde{x}}\right)^c >> 1 \tag{15}$$

holds.

5. RESULTS

In Figure 1 the GLC versus $\tilde{x}$ is shown for $\tilde{h}_S = 0.01, 0.05, 0.1$ (a-c), and $\tilde{h}_S = 0.25, 0.4, 0.5$ (d-f). For each source height two extreme Monin-Obukhov lengths are set, corresponding to $\tilde{L}_{MO}$ = 0.001, 0.099 (empty squares and triangles respectively). The second value on $\tilde{L}_{MO}$ reflects the limit imposed by the Pleim and Chang diffusivity introduced in Equation(5). The GILTT-based GLC are superimposed with the approximation of Equation(8) (dotted lines). Plots highlight that for near surface sources there is a slight mismatch between points and lines near the source position, where the horizontal gradient is most pronounced, logarithmic scales enhance such a discrepancy. As the source height increases a higher matching results, including a fair reproduction of the position where the maximum GLC occurs. As the emitting source height $\tilde{h}_S$ increases the approximated function slightly underestimate the GILTT-based maximum. Such a discrepancy reflects the fact that condition (15) is no longer satisfied. Nonetheless, through the whole range of source heights $0 < \tilde{h}_S \leq 0.5$ the function $C_{GLC}(x)$ reproduces fairly well the GILTT results.

Figures 2 and 3 show the maximum GLC and its position respectively. These are scanned through the source height $\tilde{h}_S$ and for several selected values of the turbulence parameter $\tilde{L}_{MO}$. In both figures the GILTT results (points) are superimposed on the explicit approximations. Figure 2 depicts the position where the maximum occurs, for low sources dotted GILTT results and approximated lines (Eq. (13)) show good matching regardless the turbulence regime. For higher sources a mismatch occurs and the discrepancy increases as convective turbulence reduces strength, this fact follows from the condition (15).

Turbulence dependency shows that for a fixed $\tilde{h}_S$ the strength of convection causes the for x_M to get closer to the source height. From the physics point of view this result agrees with the mixing effect of turbulence. A final remark should be made about Figure 3. Both GILTT than expression (14) confirm that the maximum GLC value depends on the source height, regardless the turbulence. Based on the expression (14) and parameters definitions (15)-(16), respectively for b, c and κ, the leading term for the maximum GLC results:

$$C_{MGLC}(x_M) \approx \tilde{h}_S^{-1} \tag{16}$$

and the exponent -1 is a lower bound. These results broaden the well known result obtained with the Gaussian approach for an unbounded ABL.

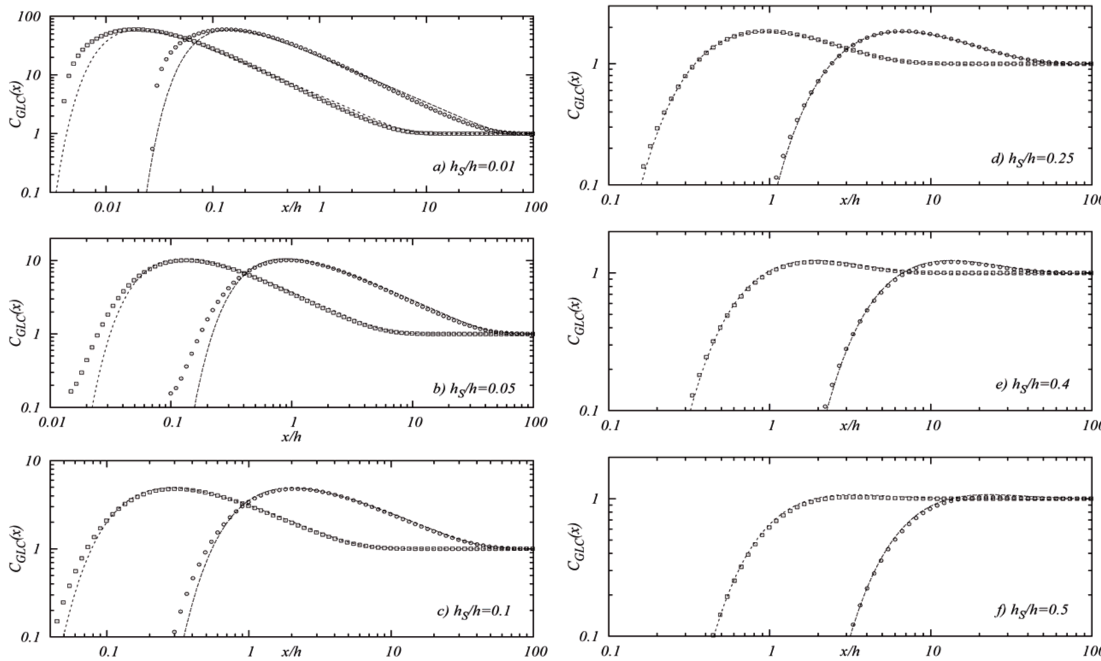

Figure 1. The GLC is plot versus $\tilde{x}$ for several source heights.

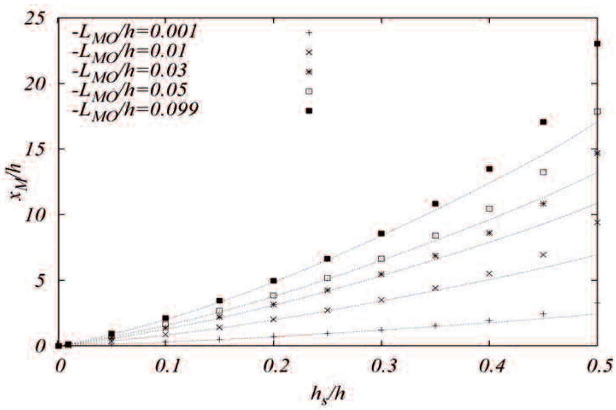

Figure 2. Plot of x_M versus $\tilde{h}_S$. Points refer to the GILTT results, dotted lines refer to Equation (13).

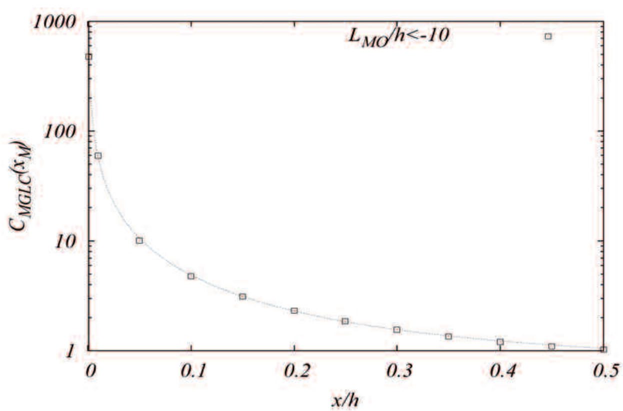

Figure 3. Plot of C_{MGLC} versus $\tilde{h}_S$. Points refer to the GILTT results, dotted lines refer to Equation (14).

6. CONCLUSIONS

The results shown in this paper have been addressed to highlight the possibility to express the GLC due to an emitting point-like source in a steady convective ABL, through a simple analytical expression. Such a function is determined after analysing the behaviour of the series expansion provided by the GILTT and whose predictions ability have been extensively demonstrated in the literature when applied to several experimental data sets. Despite the simplifications driven by focusing on the only unstable ABL regimes, the analysis allows to understand to a high extent the form of the ground level concentration.

The main progresses to be highlighted are that for a function defined as Eq.(8), within the ABL setting choice, the maximum GLC is only depending on source height, regardless the the Monin-Obukhov length. On the other hand, turbulence can still affect the position where the maximum GLC occurs. Such a result is also confirmed by the GILTT solution. A further remarkable point regards the result that for sources placed above the ABL middle level no maxima occurs as the limit become an upper bound, and the existence of a non-zero limit is one of the main properties of the two-dimensional ADE.

On the operative point of view, the expression (8) and its related features are useful as an additional tool for environmental management as well as emergency responses.

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