NUMERICAL ADVECTION EXPERIMENTS USING 2^{ND}, 4^{TH}, 6^{TH} AND 8^{TH} ORDER BOTT'S SCHEME

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Abstract: In this work different one and two dimensional numerical advection experiments using second, fourth, sixth and eighth order Bott scheme were done. The results from one dimensional numerical advection experiments with different initial functions showed that amplitude and phase errors are smallest for the 8^{th} order Bott scheme. Gibb's oscillations occur in the numerical advection experiments when there is sharp spatial gradient in the function profile. One possible way to solve this problem is to include explicit horizontal diffusion. Numerical experiments with sinus function showed that numerical diffusion for the same order Bott scheme is greater for the sinus function with smaller wavelength. Modified 4^{th} order Bott scheme produces more numerical diffusion when compared to the 6^{th} or 8^{th} order Bott scheme. Results from two dimensional numerical advection experiments showed that amplitude and phase error is smallest for the 8^{th} order Bott numerical scheme.

Keywords: *Numerical advection scheme, Bott, Air pollution modeling*

1. INTRODUCTION

The prediction of the air pollution is one of many environmental problems in modern society. Air quality models are useful tool for analyzing air pollution at different spatial and time scales. Numerical modeling of atmospheric processes by Eulerian air quality models requires the solution of the advection equation that is describing the transport of the air pollutants in the atmosphere. Numerical advection algorithms implicitly introduce numerical diffusion into air quality models (Odman, 1997). Excessive numerical diffusion can produce underestimation of peak concentrations values and overestimation of extent of concentration plume. Also, numerical advection schemes can produce non physical oscillation (Gibb's oscillation; Navarra et al, 1994). One of many numerical schemes for solving advection equation was developed by Bott (1989a,b). In Bott's numerical scheme the advective fluxes are computed by utilizing the integrated flux concept of Tremback et al. (1987). The change of a concentration in a grid point for a time step is calculated as difference at cell boundaries. The fluxes in grid points are calculated by integrating the polynomial fit (a Lagrange polynomial of even and odd order) over the neighboring grid points, normalized and then limited by upper and lower values. The Bott's numerical scheme is conservative and positively definite with small numerical diffusion. The area preserving flux form advection algorithm was extended to monotonicity by Bott (1992). Numerical results showed that amplitude and phase errors are somewhat larger in the monotone advection scheme. Two modified versions of Bott's positive-definite numerical advection schemes were developed by Easter (1993) while Bott's forward in time, positive-definite, area-preserving flux-form advection algorithm was extended to higher orders by Costa et al. (1997). The results of the two dimensional experiments (purely rotational flow and a purely deformational flow) showed that amplitude and phase-speed errors are smaller for the higher order Bott's numerical scheme. Detailed algebraic description of the Bott numerical schemes can be found in Bott (1989a,b; 1992) and Costa et al. (1997).

In this work the analysis of the numerical diffusion produced by the 2^{nd}, 4^{th}, 6^{th} and 8^{th} order Bott scheme will be done. Results of the numerical experiments are in Section 2 while in Section 3 are conclusions.

2. NUMERICAL EXPERIMENTS

One dimensional numerical experiments

Results from one dimensional numerical advection experiments using different initial functions are shown in Figure 1. Courant number was 0.4 in this experiment and results are shown after 125 time iteration in order to compare them with the results at Figure 5. by Walcek and Aleksic (1998). Time integration scheme in all numerical experiments was explicit forward Euler scheme (first order accurate). The best preservation of the amplitude and phase for the left tooth, right tooth, triangle-5, triangle-10 and step-1 functions is for the 8^{th} order Bott scheme. Strong numerical diffusion can be observed for the step-1 function for all order's of Bott scheme. But amplitude damping is smallest for the 8^{th} order scheme. Similar results for step-1 function can be found in work of Petrova et al. (2007; Fig. 2) for some other numerical schemes. Numerical diffusion for the step-10 function is produced by the Gibb's oscillations that are developing at the boundaries of the steep function (Figure 1). The ripple patterns caused by Gibb's oscillations can, for example, be found in the precipitation and cloud fields in the climate models (Navarra et al., 1994). Overall, results from this one dimensional numerical experiments showed that amplitude preservation by the 8^{th} order Bott scheme is comparable to the results from Walcek and Aleksic (1998) study.

Results from numerical advection experiments using step-50 function (width of 50 x; box function) with explicitly introduced horizontal diffusion are shown in Figure 2. Results show that advected function is smoothed by the horizontal diffusion. Also, the results show that greater smoothing of the function is produced by greater horizontal diffusion coefficient. So it is an effective way of reducing Gibbs oscillation with some degradation of the initial shape (amplitude damping). Another solution for the problem of overshooting during advection process in air quality models would be to apply monotone flux limitation algorithm (Bott, 1992).

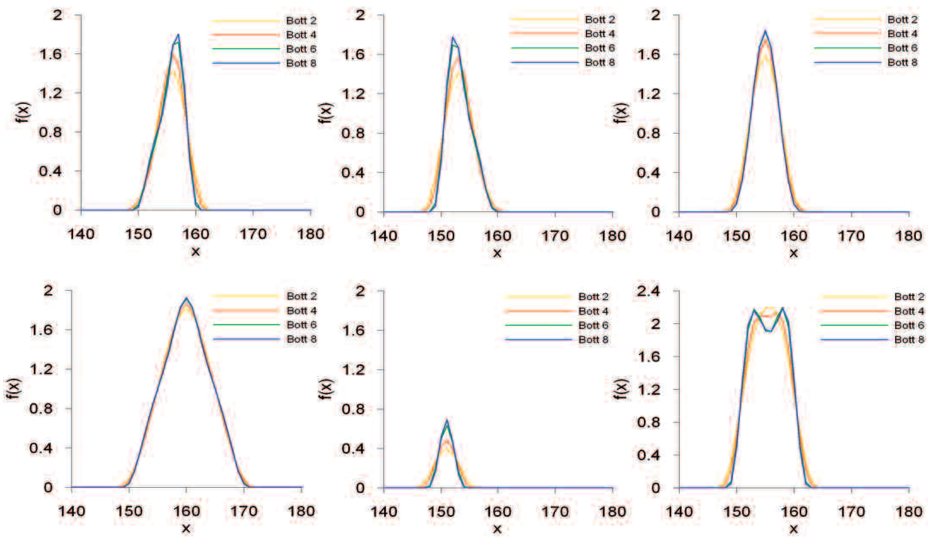

Figure 1. Numerical advection experiment results with the 2nd, 4th, 6th and 8th order Bott scheme for different initial functions (Courant number was 0.4; 125 time iterations). The functions are: left-tooth-8 (up left), right-tooth-8 (up middle), triangle-5 (up right), triangle-10 (down left), step-1 (down middle) and step-10 (down right).

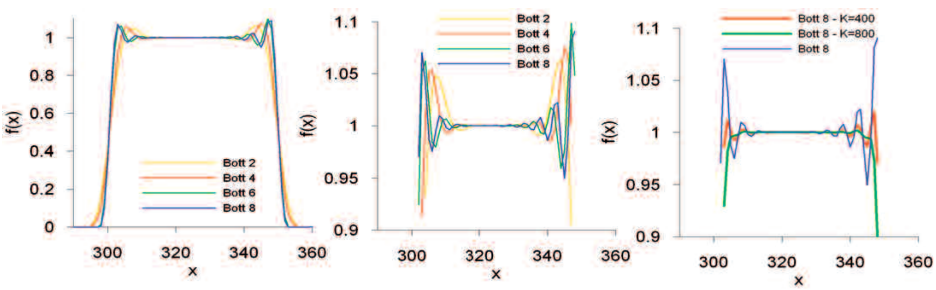

Figure 2. Numerical advection experiment results with the 2nd, 4th, 6th and 8th order Bott scheme for step-50 function (width of 50Δx; Courant number was 0.2; 999 time iterations) without horizontal diffusion (normal view, left figure; enlarged view, middle figure) and with horizontal diffusion (8th order Bott scheme; diffusion coefficients, 400 m^2s^{-1} and 800 m^2s^{-1}; right figure).

Results from the numerical advection experiments for the sinus function with different wavelengths (20Δx, 10Δx and 4Δx) are shown in Figure 3 and Figure 4. It can be seen that both amplitude and phase of the sinus function with wavelength of 20Δ x are best preserved with the 8th order Bott's scheme (Fig. 3 and Fig. 4; left. EMAX is defined in the Appendix 1). Second order numerical scheme produces strongest amplitude damping (Fig. 4). It is also interesting to observe the existence of the numerical diffusion on the edges of the sinus function for all numerical schemes (smallest for 8th order Bott's scheme). The 2nd and 4th order Bott scheme produces stronger amplitude damping of the sinus function with wavelength of 10Δx, while the 6th and 8th order schemes exhibit good overall amplitude and phase preservation (Fig. 3 and Fig. 4; middle). Finally, results for the sinus function with wavelength of 4Δx show that there is a serious numerical diffusion (amplitude and phase error) for all orders of Bott's numerical schemes, but that smallest amplitude and phase errors are for the 6th and 8th order Bott's scheme. Based on all three numerical experiments, it can be concluded that numerical diffusion is larger in advection experiments for sinus function with smaller wavelength.

In the EMEP model (Berge and Jakobsen, 1998; Olendrzynski et al, 2000) the modified version of the 4th order Bott numerical scheme is implemented (Berge and Tarrason, 1992). Different one dimensional experiments were done in order to compare the properties of the modified 4th order Bott's scheme with the original Bott's algorithm.

The amplitude of the sinus function with wavelength of 4Δx (Fig. 5. left) is strongly damped after 1000 time iterations for the original 4th and 8th order Bott's scheme. The same is valid for the 4th order modified Bott's scheme in the EMEP model. But amplitude damping is a bit smaller for the modified 4th order Bott's scheme compared to the original 4th order Bott's algorithm (Fig. 6. left). The results for the step-10 function show that there is a overshooting in all numerical schemes (Figure 5. middle). 8th order Bott scheme is producing Gibb's oscillations on the edges of the function while the original and modified 4th order Bott scheme are producing overshooting in the middle part of the box function (note: statistics EMAX is rising trough time for the modified 4th order Bott's scheme). Amplitude damping is largest for the original 4th order Bott's scheme (Fig. 6. right) in the numerical experiment for the right tooth function while best amplitude preservation is observed for the 8th order Bott numerical scheme.

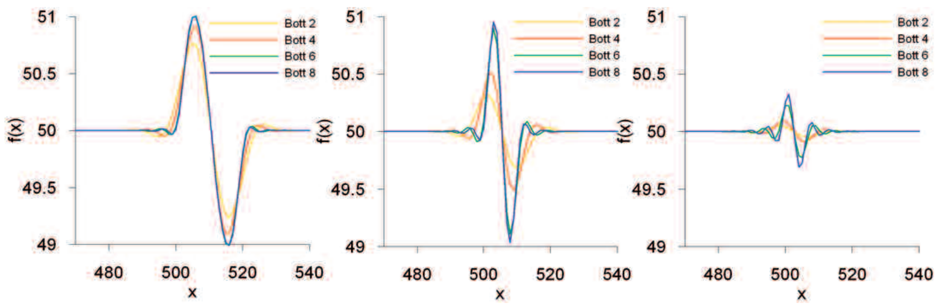

Figure 3. Numerical advection experiment results with the 2nd, 4th, 6th and 8th order Bott scheme for sinus-20 (wavelength of 20Δx; left), sinus-10 (wavelength of 10Δx; middle) and sinus-4 (wavelength of 4Δx; right) function. Courant number was 0.4 (999 time iterations).

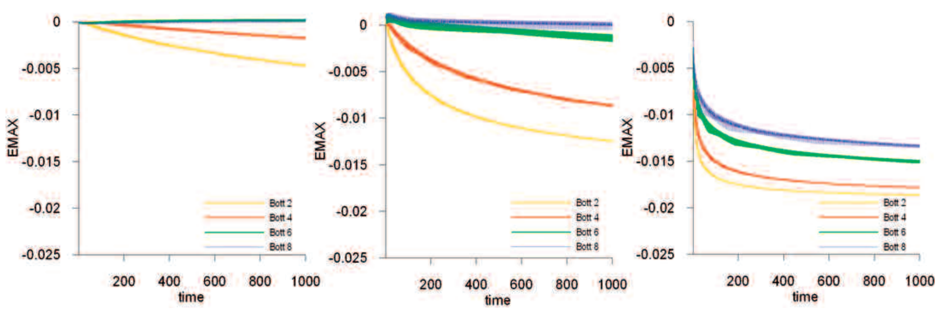

Figure 4. Time change of statistical measure EMAX with the 2nd, 4th, 6th and 8th order Bott's scheme for sinus-20 (wavelength of 20Δx; left), sinus-10 (wavelength of 10Δx; middle) and sinus-4 (wavelength of 4Δx; right) function. Courant number was 0.4 (999 time iterations).

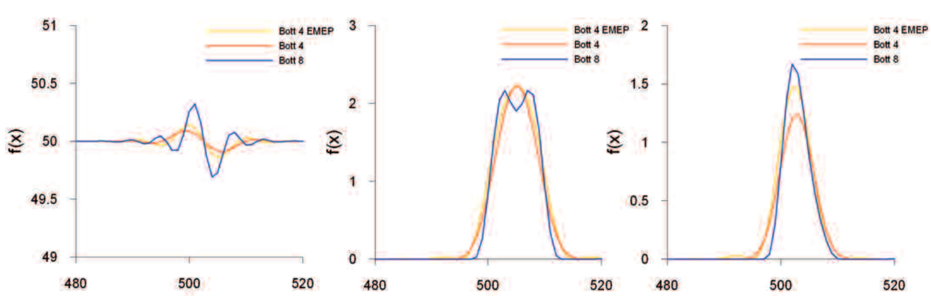

Figure 5. Numerical advection experiment results with the original 4th, modified 4th and original 8th order Bott scheme for sinus-4 (wavelength of 20Δx; left), step-10 (middle) and right tooth-8 (right) function. Courant number was 0.4 (999 time iterations).

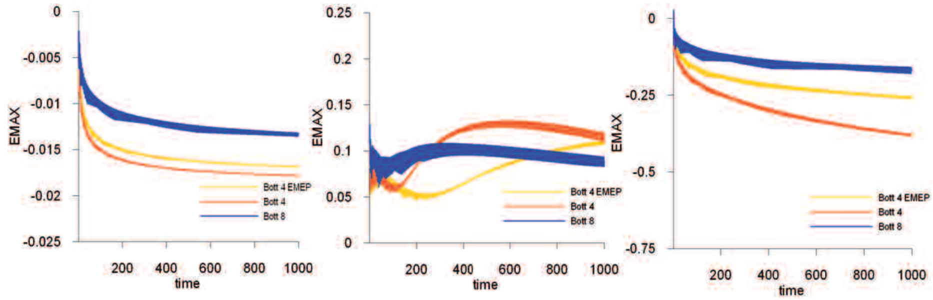

Figure 6. Time change of statistical measure EMAX with the original 4th, modified 4th and original 8th order Bott's scheme for sinus-4 (wavelength of 20 Δx; left), step-10 (middle) and right tooth-8 (right) function. Courant number was 0.4 (999 time iterations).

Two dimensional numerical experiments

Different two dimensional experiments (rotation of the cone; front; cube; point source, deformational flow field test) similar to other studies (Bott, 1989; Costa and Sampaio, 1997; Petrova et al, 2007) were done in order to analyze numerical diffusion produced by 2nd, 4th, 6th and 8th Bott numerical scheme. Only results from the modified numerical experiment (Appendix 2) designed by the Durran (1999; advection of the passive tracer, section 5.7.4.) will be shown. Numerical advection experiment in two dimensions was done by successive application of one dimensional algorithm. In other words, advected quantities are first updated doing x-direction advection using 'u' velocities. These updated values are then advected in the y-direction using 'v' velocities. The order of advection (x then y, y then x) was changed every time step (Strange, 1968). In that way, a 2nd order accuracy is achieved (Skamarock, 2006).
Results of numerical experiment for the 8th order Bott's scheme during time integration are shown in Figure 7. (a-e). The final positions are shown at Figure 7. (f-i). The phase error at the final time step is smallest for the 8th order Bott's scheme while it is largest for the 2nd order Bott's scheme (Fig 7a; Fig. 7f; Fig 7i). Beside phase error the amplitude error also differs for different orders of Bott scheme. Best amplitude preservation can be observed for the 8th order Bott's scheme (Fig. 7j) while amplitude damping is largest for the 2nd order Bott scheme. From this results it is obvious that numerical diffusion is smallest for the 8th order Bott's scheme.

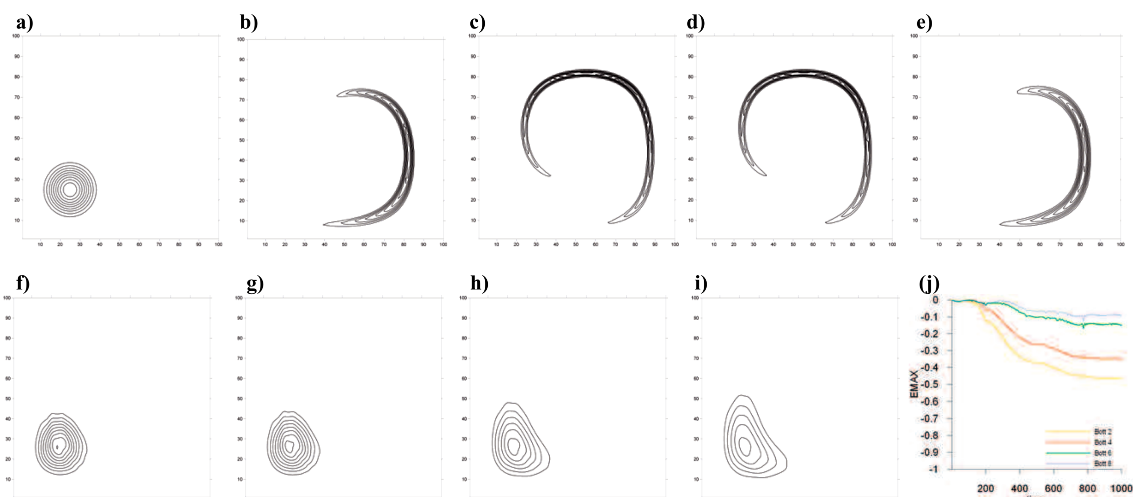

Figure 7. Two dimensional numerical experiment results with the 8th order Bott scheme (a) t=0ΔT. (b) t=200ΔT. (c) t=400ΔT. (d) t=600ΔT. (e) t=800ΔT. (f) t=1000ΔT. (g) t=1000ΔT; 6th order Bott. (h) t=1000ΔT; 4th order Bott. (i) t=1000 ΔT; 2nd order Bott. (j) Time change of statistical measure EMAX with the 2nd, 4th, 6th and 8th order Bott scheme in the two dimensional numerical experiment (Appendix 2).

3. CONCLUSIONS

In this work analysis of numerical diffusion introduced by 2nd, 4th, 6th and 8th order Bott scheme in the advection equation was done. Results from one dimensional experiments showed that smallest numerical diffusion is produced by the 8th order Bott's scheme. Gibb's oscillations occur in the numerical advection experiments when there are sharp spatial gradients in the function profile. One possible way to solve this problem is to explicitly include physical diffusion in the advection equation. Numerical experiments with sinus functions showed that stronger amplitude damping is occurring during transport of the sinus functions with shorter wavelengths. Modified 4th order Bott scheme have better amplitude preservation then original 4th order scheme. But smallest numerical diffusion is produced by 8th order Bott scheme. Numerical experiments in two dimensions also confirmed that most precise solution of the numerical advection equation can be achieved by the 8th order Bott numerical scheme.

REFERENCES

Berge E. and H.A. Jakobsen, 1998: A regional scale multi-layer model for the calculation of long-term transport and deposition of air pollution in Europe. *Tellus,* **50B**. 205-223.

Bott, A., 1989a: A positive definite advection scheme obtained by nonlinear renormalization of the advective fluxes. *Monthly Weather Review,* **117**. 1006-1015.

Bott, A., 1989b: Reply. *Monthly Weather Review,* **117**. 2633-2636.

Bott, A., 1992: Monotone Flux Limitation in the Area preserving flux form advection algorithm, *Monthly Weather Review.* **120**. 2592:2602.

Costa, A.A. and A.J.C. Sampaio, 1997: Bott's area-preserving flux-form advection algorithm: extension to higher orders and additional tests. *Monthly Weather Review,* **125**. 1983:1989.

Durran, D., 1999: Numerical Methods for Wave Equations in Geophysical Fluid Dynamics. New York: Springer – Verlag, ISBN 0-387-98376-7, 465 pp.

Easter, R.C., 1993: Two modified versions of Bott's positive-definite numerical advection scheme. *Monthly Weather Review,* **121**. 297-304.

Navarra, A., W.F. Stern and K. Miyakoda, 1994: Reduction of the Gibbs oscillation in Spectral Model Simulations. *Journal of Climate,* **7**. 1169-1183.

Odman, M.T., 1997: A quantitative analysis of numerical diffusion introduced by advection algorithms in air quality models. *Atmospheric Environmen,.* **31**. 1933-1940.

Olendrzynski K., E. Berge and J. Bartnicki, 2000: EMEP Eulerian acid deposition model and its applications. *Journal of Operational Research,* **122**. 426-439.

Petrova, S., H. Kirova, D. Syrakov and M. Prodanova, 2007: Some fast variants of TRAP scheme for solving advection equation-comparison with other schemes. *Computers and Mathematics with Applications*. doi: 10.1016/j.camwa.2007.11.001.

Skamarock, W.C., 2006. Positive-Definite and Monotonic Limiters for Unrestricted-Time Step Transport Schemes. *Monthly Weather Review,* **134.** 2241-2250**.**

Strange, G., 1968: On the construction and composition of difference schemes. *SIAM J. Numer. Anal.*, **5.** 506-517.

Tremback, C.J., J.Powell, W.R. Cotton and R.A. Pielke, 1987: The forward in time upstream advection scheme: Extension to higher orders. *Monthly Weather Review,* **115**. 540-555.

Walcek, C.J. and N.M. Aleksic, 1998: A simple but accurate mass conservative peak-preserving, mixing ratio bounded advection algorithm with FORTRAN code. *Atmospheric Environment,* **32**. 3863-3880.

APPENDIX 1:

Statistical measure EMAX:

$$\mathrm{EMAX}\,(t) = \frac{\max\,(f(x_i))_t}{\max\,(f(x_i))_0} - 1$$

APPENDIX 2:

Numerical experiment was very similar to the one by D. Durran (1999, Section 5.7.4). Instead constant 4.0 (algebraic expression for the r(x,y)) we used constant 6.0 because 8th order Bott's numerical scheme requires constant boundary conditions during all time integration steps (note: the initial function in our numerical test was more narrow than in the original experiment).
Nx = 100; number of points in x-direction. Ny = 100; number of points in y-direction. Nt = 1000; number of time integration steps. Δt = 0.005 s; increment in time domain. Δx = 0.01 m; increment in space domain. Δy = 0.01 m; increment in space domain. T = 5 s; time integration interval.

$$\varphi(x,y,t=0) = \frac{1}{2}\left[1+\cos(\pi r)\right]$$

$$r(x,y) = \min\left\{1,\ 6\left[\left(x-\frac{1}{4}\right)^2+\left(y-\frac{1}{4}\right)^2\right]^{\frac{1}{2}}\right\}$$

$$u(x,y,t) = \sin^2(\pi x)\sin(2\pi y)\cos(\frac{\pi t}{5})$$

$$v(x,y,t) = -\sin^2(\pi y)\sin(2\pi x)\cos(\frac{\pi t}{5})$$